Worlds Apart:
On the Possibility of An Actual Infinity

§1 The *Kalam* Cosmological Argument and the Finitude of the Past

Cosmological arguments attempt to prove the existence of God by appeal to the necessity of a first cause. Schematically, a cosmological argument will thus appear as:

(1) All contingent beings have a cause of existence.

(2) There can be no infinite causal chains.

(3) Therefore, there must be some non-contingent First Cause.

Cosmological arguments come in two species, depending on their justification of the second premiss. Non-temporal cosmological arguments, such as those of Aristotle and Aquinas, view causation as requiring explanatory or conceptual priority, and thus insist that there can be no infinite regresses in such priority. Temporal cosmological arguments, also called *kalam* cosmological arguments due to their historical roots in Islamic *kalam* philosophers such as Abu Yusuf Ya'qub b. Ishaq al-Kindi and Abu Ali al-Hussain ibn Sina, view causation as requiring temporal priority, and thus insist that there can be no infinite temporal regresses.[1]

The *kalam* cosmological argument thus requires some supporting argument showing the incoherence of an infinite temporal regress of causally related events. William Lane Craig, in "The Finitude of the Past and the Existence of God"[2], attempts to provide such an argument:

(4) An actual infinite cannot exist.

(5) An infinite temporal regress of events is an actual infinite.

(6) Therefore an infinite temporal regress of events cannot exist. (9)

I will not be concerned here with the general status of cosmological arguments, *kalam* or otherwise, or with contesting Craig's assumption that an infinite past would (unlike an infinite future) constitute a problematic actual infinity. I am rather concerned with Craig's general working principle, embodied in (4) above, that actual infinities are impossible.

Craig, of course, is not alone in denying the possibility of the actually infinite. Resistance to such infinities is at least as old as Aristotle (Physics 3.5.204b1 – 206a8), and, as Craig rightly points out, persists through much of modern (i.e., post-scholastic, pre-twentieth-century) philosophy. Contemporary opponents of the actually infinite (such

[1] See Craig (1979) for extensive discussion of the *kalam* cosmological argument, and Craig (1980) for a general overview of cosmological arguments.
[2] Craig (1993). All references will be to this paper unless otherwise noted.

as Craig), however, have a new hurdle to overcome – the widespread acceptance of actual infinities in mathematics, due largely to the work of Georg Cantor.

Faced with a well-founded mathematics of the transfinite, anti-actualists must either oppose the conceptual foundations of the mathematics -- cast mathematicians out of "Cantor's paradise" -- or attempt to deny its relevance to the philosophical issue. While Craig flirts with the first course (16-22), what interests me more, and what will be my focus here, is his assertion that "while the actual infinite may be a fruitful and consistent concept in the mathematical realm, it cannot be translated from the mathematical world into the real world, for this would involve counter-intuitive absurdities" (9).[3] Craig thus appears to endorse the canonical mathematical view that there can be collection with infinite cardinalities, while simultaneously denying that any collection of *real* -- that is, non-mathematical -- objects can have such infinite cardinality. It is this disparity between the condition of the "mathematical world" and the "real world" that I want to investigate.

§2 The Mathematical World and the Real World

What sense are we to make of the suggestion that actual infinities are mathematically possible, but cannot manifest in the real (i.e., I take it, physical) world? Perhaps Craig means that something analogous to the following: a four-dimensional "hypersphere" -- the collection of all points within some finite distance of some distinguished point in the space $\mathfrak{R}^4$ -- is mathematically possible; such objects do exist "in the mathematical world". However, there can be no hypersphere "in the real world". Thus while a hypersphere "may be a fruitful and consistent concept in the mathematical realm, it cannot be translated from the mathematical world into the real world".

The disparity between the possibility of a physical and a mathematical hypersphere, however, seems to differ from Craig's proposed difference between physical and mathematical possibility of an actual infinity in two interrelated ways.[4] First, the impossibility of a physical hypersphere is presumably a relatively weak impossibility.

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[3] Craig argues against the possibility of actual infinities in Craig (1979), Craig (1985), Craig (1991), Craig (1993), Craig (1993a), and Craig (1995); I will focus here primarily on Craig (1993). Craig is not alone among recent philosophers in arguing against the physical possibility of actual infinities; see also Conway (1974), Conway (1984), Huby (1971), Whitrow (1966), and Whitrow (1978). These other authors generally share Craig's desire to square a denial of actual infinities with an acceptance of Cantorian mathematics, but since Craig is most explicit in formulating the 'two worlds' principle, I focus primarily on his work.

[4] The possibility of a physical actual infinity, of course, needs to be distinguished from the physical possibility of an actual infinity. Craig's claim, I take it, is that an actually infinite collection of physical objects in impossible simpliciter, not merely forbidden by physical law. Oppy (1991) considers and rejects the possibility that Craig means that actual physical infinities are metaphysically impossible, suggesting instead that Craig intends to assert that they are merely physically impossible. The style of Craig's argument, however, from the "absurdities" of physical actual infinities, coupled with his failure to *appeal* to any physical law blocking such infinities, suggests to me that he intends to assert their impossibility in (at least) all metaphysically possible physical worlds.

While it is certainly true, and perhaps even physically necessary, that space is three-dimensional, there is no apparent reason to believe that it is absolutely, or metaphysically, necessary that it be so. The impossibility of a physical hypersphere is thus similar in kind to the inapplicability of Euclidean geometry to a physical space -- a contingent result of the particular physical laws governing our world.

Second, there is a clear explanation for the impossibility of a physical hypersphere. That impossibility derives from features of the environment -- here three-dimensional space -- in which we are attempting (and failing) to instantiate the particular mathematical concept. Thus assume there are some ultimate, indivisible building blocks of the universe -- call them 'atoms'. Then, while the mathematical concept '3 1/2' would fail of instantiation in atoms, since we could not have 3 1/2 atoms, there would be a clear explanation for the failure. Constitutive properties of the atoms, namely their indivisibility, would explain the failure of instantiation, and we would expect that instantiation of the mathematical concept '3 1/2' would be possible in other kinds of stuff lacking some inhibiting feature.

Given these considerations, the following principle of mathematical application seems prima facie plausible:

> (MA) Given any mathematical claim, it is metaphysically possible that there be some world, or some portion of a world, which serves as a model for that claim.

Thus, for example, given the mathematical claim that every natural number has a unique successor, it is metaphysically possible that there be some world containing a collection of (physical) objects and some relation among these objects, such that the mathematical claim, interpreted so as to take those objects as the extension of the predicate 'is a natural number' and that relation as the intension of the predicate 'is the successor of', is true with respect to that world.

Craig, in denying the possibility of a physical actual infinity, denies one particular instance of the principle (MA), and does so by endorsing what I will call the 'Two Worlds Principle':

> (2W) There is no prima facie reason to believe that mathematical claims are realizable in physical reality.

The Two Worlds Principle lies behind Craig's willingness to assert that Cantor's introduction of actual infinities into the everyday toolbox of mathematicians is of no relevance to the question of the possibility of 'real' actual infinities.

I am disturbed by the distance Craig, via the Two Worlds Principle, places between mathematics and physical reality. While it may at times be true that certain mathematical phenomena of necessity fail to manifest in the physical world, we in general want an explanation of such a failure of fit. Such an explanation, when it is forthcoming, should be of the form of a specification of some essential properties of matter which make matter in principle incapable of instantiating a particular mathematical structure. It is difficult to see how a satisfactory account of the applicability of mathematics to physical reality will prove possible if mathematical results can mysteriously and arbitrarily fail to hold, even counterfactually, of physical reality.

Craig's arguments against actual infinities come in three large categories: those derived from his consideration of the behaviour of a speculative "infinite library", embodying an actually infinite collection in the real world; those derived from his consideration of the so-called "Tristram Shandy" paradox; and those derived from considerations against the possibility of creating an actual infinity via successive addition. My goal in this paper is to examine the resulting purported "counter-intuitive absurdities" that arise out of the supposition of an actual infinity of physical objects, and to show (a) that if one accepts the validity of the actually infinite in the mathematical realm there is little room left for identification of some model-blocking property of the physical satisfying the above desideratum, and (b) that in each case, much of the appearance of absurdity, whether in the physical or the mathematical realm, vanishes upon reflection. Upon examination of Craig's particular objections to physical realization of actual infinities, we will see how unmotivated his endorsement of the Two Worlds Principle is.

§3 Problems in the Infinite Library

Craig's first battery of arguments against the physical possibility of an actual infinity revolve around a series of thought experiments regarding a hypothetical "infinite library", which we can take to consist of a single shelf, extending infinitely in one direction, filled completely with equi-sized books. Craig draws out a number of putative absurdities from the infinite library, all of which in the end derive from the well-known oddities of one-to-one correspondence between infinite sets. Thus, for example, the prime numbers can be placed in a one-to-one correspondence with the integers, even though there are, in some intuitive sense, "more" integers than there are prime numbers, simply because there are integers which are not prime, but no primes which are not integers. That some sets can be placed in a one-to-one correspondence with a proper subset of themselves is, admittedly, a startling fact when first learned.[5] But this oddity, mathematicians have long since conceded, is not indicative of any *problem* – it's just a failure of ideas about the finite to extend to the infinite. The proof of the existences of such correspondences is beyond reproach -- indeed, it is common simply to define the concept of an infinite set as a set which can be placed in a one-to-one correspondence with some proper subset of itself -- and since Craig wants to endorse the mathematical correctness of work in Cantor's paradise, we can safely assume that he also accepts the existence of such correspondences. Once one works with infinite sets any reasonable amount, the one-time oddity comes to have a rightful feel of inevitability.

Since Craig does not contest the correctness of the mathematical results, he must look elsewhere to block the physical possibility of an infinite collection. He thus argues that the mathematical results become *absurd* when translated into the physical world. Craig sets out three putative absurdities of the infinite library. Let us consider each in turn.

First Absurdity: Craig says:

> Suppose...that there were only two colours of book, black and red, and that every other book was the same color...Would we believe someone who told us that the number of red books in the library is the same as the number of black books plus the number of red books? (12)

Keep in mind here that Craig continues to accept all relevant mathematical claims -- just not their applicability to the physical world. So, in particular, he accepts corresponding mathematical claims, such as the claim that the number of even integers is the same as the number of even integers plus the number of odd integers. Somehow, however, this fact is unproblematic while the equinumerosity of the red with the *black or red* books is "absurd". But how could this possibly be the case? After all, we could simply inscribe the books consecutively with the positive integers, so that each red book received an even integer and each black book received an odd integer. Ex hypothesis, Craig accepts that the number of even integers is the same as the number of even or odd integers; how then could he deny that the number of books marked with even integers would be the same as the number of books marked with even or odd integers? The mathematical facts here map directly onto the physical facts; there is no room for acceptance of one and rejection of the other.

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[5] Although, strangely, it is equally puzzling to discover that the real numbers are of a different cardinality of infinity from the integers – even though, as a matter of straightforward logical necessity, one of these two strange discoveries must be true.

Moreover, why would we be inclined to doubt that the number of red books was the same as the number of black or red books? Given that the number of red books is infinite and that the number of black or red books is infinite, one would think that there would be an intuitive appeal to the claim that the number of each was the same -- namely, infinite. The strangeness of Cantor's proof that the real numbers are of a greater cardinality than the natural numbers derives precisely from its overthrow of the appealing, albeit false, view that any two infinite collections are equinumerous. Of course, to say that the number of red books is the same as the number of black or red books is not to deny that there are black or red books which are not red books.

Second Absurdity: Crucial to the plausibility of the claim that the number of red books is the same as the number of red or black books is the (mathematically provable) claim that one can add a new object to an infinite collection without increasing the number of that collection. Craig thus next attempts to show that the process of adding to a physically real infinite collection leads to further absurdities:

> Suppose further that each book in the library has a number printed on its spine so as to create a one-to-one correspondence with the natural numbers. Because the collection is actually infinite, this means that every possible natural number is printed on some book. Therefore, it would be impossible to add another book to the library. For what would be the number of the new book? Clearly there is no number available to assign it. (12-13)

There are numerous flaws with this argument. First, of course, the physical possibility of adding a book to the library in no way depends on there being a number available to assign to the book. It is no unusual feat to add a book to a bookshelf without inscribing any number at all on its spine. The absurdity, then, must derive not from our inability to add a book, but rather our inability to number that book.

But, in fact, we can number the book. Craig considers and rejects two proposals for the numbering of the new book; I will show that both are perfectly coherent. Craig rejects the possibility that we appeal to the (as yet unused) transfinite ordinals for the numbering of the new book, saying "[the new book] could not be $\omega + 1$, for this has the same cardinal number as ω, and we need a new cardinal number for this book" (13). This claim, however, is a simple confusion. We do not need a new cardinal number for the new book, because the new book has not increased the cardinality of the collection of books. A new ordinal, like $\omega+1$, will suffice for the extension of the numbering scheme to the new book.

However, the fact that all natural numbers have already been used creates no need to appeal to transfinite ordinals to accommodate the new book. Craig, in fact, makes the correct counting principle explicit at one point, when he says "there would be no number for the new book -- but this is absurd, since entities that exist in reality can be numbered" (13). The principle that entities can be numbered is certainly correct, but the modal force of it is crucial. What is true is that any collection of entities can be put into a one-to-one correspondence with some initial segment of the ordinals (finite or transfinite). But clearly we can put the augmented collection of books into such a correspondence -- simply label the first book '1', and increase the label of every other book by 1. Imagine an actually infinite set E of all even numbers, and a function f mapping the whole numbers onto the even numbers. Would we want to say that it was impossible to add another object to the set E, because there would be no numbers left to correlate with the new object? Of course not. We can in fact add a new object to E, and there will be another function f* mapping the whole numbers onto the augmented E*.

Craig, rather peculiarly, says in response to this renumbering proposal:

> This is perfectly successful in the mathematical realm, since we accommodate the new number by increasing all the others out to infinity. But in the real world this could not be done. For an actual infinity of objects already exists that completely exhausts the natural number system -- every possible number has been instantiated in reality on the spine of a book. Therefore, book 1 could not be called book 2, and book 2 be called book 3, and so on, to infinity. ... In an actual infinite, all the members exist in a determinate complete whole, and such a re-count would necessitate the creation of a new number. (13)

Again here we see puzzling claims about disparities between what is possible in "the mathematical realm" and what is possible in "the real world". Consider again E and its enumerating function f. f does indeed 'completely exhaust the natural number system' – every natural number is given an assignment by f. But this fact about f is utterly irrelevant to the question of whether there exist other functions f* which map the natural numbers into larger sets than E – and, in point of fact, it is quite easily proven that there are such function. What, then, is to prevent us from writing the results of some such f* onto our books? If Craig accepts that the set of natural numbers can be placed into a one-to-one correspondence with that set union the singleton set {-1}, then it is completely opaque why the set of natural numbers could not similarly be placed in a one-to-one correspondence with the original set of books numbered by the natural numbers union the singleton set of the new book. Of course, in the actual physical collection of books, each number is already inscribed on some book. But since the numbering claim is just that the books *could be* numbered, their actual numbering is irrelevant. There is, in short, no room for separating the possibility of numbering a superset of the natural numbers by the natural numbers from the possibility of numbering a superset of an actually countably infinite collection of physical objects; and, moreover, no reason to doubt that first, purely mathematical possibility.

Third Absurdity: Craig suggests that further absurdities result when we consider the results of removing books from the library. Suppose, for example, we remove all the black books from the shelf. Could we believe that there were just as many books on the shelf as were there before the removal? In particular, could we believe that, were we to "close the gaps between the books, all the shelves would remain full?" (15). Were that the case, where would the room for the extra books come from?

Given that Craig does not dispute the correctness of purely mathematical results, he *a fortiori* does not dispute that there is a measure-preserving translating function mapping the collection of all intervals [2n,2n+1] onto the entire real line. Such a function, of course, achieves the mathematical analog of compressing all of the remaining red books to fill the shelf. So if there is an absurdity here, it is an absurdity which depends crucially on the physicality of the books. But what would that new source of absurdity be? The intuitive puzzle is that we are able, just by shuffling things around, to change the amount of free room available. But this holds true in the mathematical case just as much as in the physical case.

Craig suggests that the source of the absurdity here lies in a disparity between the mathematical and physical realms when it comes to subtraction:

> It may be said that inverse operations cannot be performed with the transfinite numbers -- but this qualification applies to the mathematical world only, not the real world. While we may correct the mathematician who attempts inverse operations with transfinite numbers, we cannot in the real world prevent people from checking out what books they please from our library. (15)

Thus, for example, Craig claims a further absurdity that we can remove an infinite number of books from the library in one way -- by removing all odd-numbered books -- and have an infinite number of books remaining in the library, but remove an infinite number of books in another way -- by removing all books numbered '7' or greater -- and have only a finite number of books remaining. Oppy (1995) correctly points out that we can in fact define inverse operations on transfinite ordinal numbers. But this response misses the real source of confusion here. Again Craig claims a mysterious mismatch between the "mathematical" world and the "real" world. Indeed, people may check out what books they will. But the mathematician is equally free to remove what elements he will from a given infinite set. There is no important mismatch between the two worlds here; a difficulty for physical realizations of transfinite operations will translate immediately into a difficulty for the mathematical analysis of such operations.

However, there is no such difficulty, in either world. To say that subtraction is, standardly, undefined on the transfinite cardinals is not to say that some number of objects cannot be taken away from some collection of objects. It is rather to say that the process of taking away elements from a set does not lead naturally to a definition of a formally correct subtraction operation. Consider here how cardinal addition is standardly defined (see, e.g., Kunen 1980). Given any two sets X and Y, define their disjoint union as:

$$X \times \{0\} \cup Y \times \{1\}$$

We can now define the sum of the cardinality of X with the cardinality of Y as the cardinality of the disjoint union of X and Y. Proving the formal correctness of this definition then involves establishing the following theorem:

Given sets X_1, X_2, Y_1, Y_2 such that:

$$|X_1| = |X_2|$$

$$|Y_1| = |Y_2|$$

we have:

$$|X_1 \times \{0\} \cup Y_1 \times \{1\}| = |X_2 \times \{0\} \cup Y_2 \times \{1\}|$$

Cardinal addition is thus a derivative operation -- the sum of two cardinals is the cardinality of the disjoint union of two sets of the appropriate cardinality -- and to show its formal correctness we must show that it derives in the same way from all instances, and thus show that no matter which two sets of the appropriate cardinality we pick, their disjoint union has the same cardinality. No such theorem is available for cardinal subtraction. Two sets of the same cardinality can both have removed from them subsets of the same cardinality, and sets of differing cardinality will result. Thus we cannot take cardinal subtraction to be the uniform result of set subtraction among sets of like cardinality.

If Craig accepts the mathematical results, it is unclear where he wishes to insert the wedge between the mathematical and physical worlds. In both worlds, one can remove entities arbitrarily from any given collection; in both worlds, two ways of removing the same number of entities from a given collection may result in different numbers of remaining entities. A surprising result, perhaps -- but if an absurd one, then every bit as absurd in the mathematical as in the physical realm.

§4 The Principle of Correspondence and Euclid's Maxim

Craig identifies as the root of the absurdities of the physically actually infinite what he sees as an unjustified assumption by mathematicians that Cantor's Principle of Correspondence is valid in the transfinite realm. Cantor's Principle of Correspondence is the claim that any two sets which can be placed in a one-to-one correlation are equivalent; this principle, of course, underlies Frege's definition of number in the *Grundlagen*. Craig points out that whatever intuitive appeal may be possessed by the Principle of Correspondence is matched by the appeal of Euclid's Maxim -- the principle that the whole is greater than any of its parts. In the finite realm, of course, the Principle of Correspondence and Euclid's Maxim are in perfect agreement, but, says Craig, come the infinite realm and the two are in disagreement:

> Moreover, it is clear that they cannot both be true for infinite collections, since they are, in this case, contradictory principles: one asserts that the whole is greater than a part, while the other asserts that the whole is not greater than a part. But which principle is to be sacrificed? Both seem intuitively obvious principles in themselves, and both seem to result in counter-intuitive situations when applied to the actual infinite. The most reasonable approach to the matter seems to be to regard both principles as valid in reality and the existence of an actual infinite as impossible. (24)

It is difficult to know how to square this passage with Craig's avowed acceptance of the mere mathematical possibility of actual infinities. In the mathematical realm we cannot regard the existence of an actual infinite as impossible and thus must, it would seem, choose between the two principles. Once we have made a choice, furthermore, it would seem that our grounds for choice would likely carry over to the physical realm as well, and the argument against the physical possibility of an actual infinity would vanish. Craig's opposition between the Principle of Correspondence and Euclid's Maxim, in any case, strikes me as a red herring. Surely the more reasonable approach is to say that both the Principle of Correspondence and Euclid's Maxim encapsulate notions of 'more' -- call them '$more_C$' and '$more_E$' -- and that transfinite mathematics have revealed to us that the two notions of 'more' are not coextensive.

Craig consistently equivocates between '$more_C$' and '$more_E$'. Thus consider the following claim, made during his discussion of the possibility of adding a new book to the library:

> But suppose we could add to the infinite collection of books. The new book would have the ordinal $\omega+1$. And yet our collection of books has not increased by a single book. But how can this be? We put the book on the shelf: there is one more book in the collection; we take it off the shelf: there is one less book in the collection. (14)

Contra Craig, the most hardened advocate of the actually infinite need not deny that the collection of books has been increased by a single book, so long as it is understood that the claim is that the collection has been increased_E. A book is there that was not there before; the current collection of books is a superset of the prior collection. Nevertheless, the collection has not been increased_C. There is no gain in cardinality; the current collection can be placed in a one-to-one correspondence with the prior collection.

How *could* it be the case that adding one object to an actually infinite collection in the *physical* world entails an increase in cardinality, even when there is no corresponding increase in the *mathematical* world? Cardinality is, after all, a mathematical notion. To say that a collection, even a physical collection, has a given cardinality, is by definition to make a statement about what one-to-one correspondences exist between that collection and certain set-theoretic constructs. Neither the set-theoretic constructs nor the correspondences are, in any ordinary sense of the word, physical, so any mysterious deficiency of the physical cannot be relevant to questions about the size of cardinalities.

The plausibility of Euclid's Maxim can be taken to rest on the prior plausibility of the Principle of Correspondence. Why is the whole greater than any of its parts? Well, if you imagine all of the objects in the part in a line, and all of the objects in the whole in another adjacent line, with the initial segment of the 'whole' line identical to the totality of the 'part' line, there will be 'stuff left over' at the end of the whole line. Thus the natural method of creating a correspondence between part and whole -- via the identity mapping -- will of necessity fail to find a one-to-one correspondence between part and whole. Correspondence thus drives Euclid's Maxim, but fails to motivate it adequately when we reach the transfinite realm, since here the failure of one method of one-to-one correlation to demonstrate equivalence does not entail the failure of all methods.

It is this last peculiarity of the infinite that, it seems to me, is genuinely at the root of Craig's worries. Given two infinite sets X and Y, there may well be a one-to-one mapping of X properly into Y, a one-to-one mapping of Y properly into X, and a one-to-one mapping of X onto Y. Thus the apparent 'size' of an infinite set can be changed just by rearranging it -- 'gaps' in the infinite library can be closed; checking out an infinite collection of books sometimes leaves the library impoverished and sometimes leaves it infinitely well stocked; etc.

If this peculiarity of the infinite is indeed at the root of Craig's worries, then, as Gödel (1964) points out, Craig has reason to favor the Principle of Correspondence over Euclid's Maxim as a measure of size in the transfinite. Gödel claims that precisely because the existence of one-to-one correlation between two sets is permutation-invariant, while part-whole relations are not permutation-invariant, we are driven to accept the Principle of Correspondence as the measure of number in the transfinite realm.

Furthermore, this peculiarity of the infinite is *mathematically* well-established. No one doubts that there are one-to-one mappings of the even integers to the integers, some of which are into the integers and some of which are onto. Where, then, is the room for doubting that such varying correlations among infinite collections of physical objects are possible? And again, there seems no room for doubting the coherence of the strictly mathematical results. That results of one-to-one correlations should vary in this way is a predictable consequence of the very notion of infinity. After all, if a collection is infinite, one can set aside one element of it and still have infinitely many elements left. If one sets in this manner sets aside one element and then maps the remainder one-to-one onto another infinite set, then one will find one's starting set to be 'larger' than the target set. If one does not set aside an element, then one will find the starting and target sets to be 'of the same size'.

§5 Tristram Shandy

Craig provides another line of attack against the physical possibility of actual infinities through his consideration of the 'Tristram Shandy' paradox. In its original form, due to Russell (1937), this paradox borrows Sterne's conceit of Tristram Shandy writing his own autobiography at a rate of one day transcribed per year of writing, and notes that, although Shandy will fall further and further behind in his writing, he will, if he writes infinitely, complete the autobiography. Craig castigates Russell here for confusing actual and possible infinities, holding that all that is true is that Shandy will eventually, if he writes long enough, transcribe the events of any given day, but denying that there is any point in time at which Shandy completes the autobiography.

Craig's objection to Russell is, of course, a confusion between what happens during an actually infinite collection of days and what has happened at the first day after an actually infinite collection of days. Thus consider Whitrow (1978)'s claim:

> If Tristram Shandy actually succeeded in living for an infinite number of years, then the time would eventually arrive, as Russell said, when all the days of his life would have been written about (42).

In fact, Russellis considerably more cautious than this. He asserts only that if Tristram Shandy lives infinitely long, all of his life will be written in his autobiography, not that there is any time at which all of his life will have been written about. Russell's point is that if we have some function f mapping the day of writing into the day being written about:

$$f(x) = int(x/365)$$

(where 'int' maps any real to the greatest integer less than it), then if we consider the range of f over the domain of the natural numbers, we will discover that range also to be the natural numbers. If Shandy writes for every day in an actually infinite sequence, he will write about every day. This is not, however, to say that the function f can in any natural way be extended to a function on a set of ordinal type $\omega+1$. Thus we need not consider questions about what day Shandy is writing about on the $\omega^{\underline{th}}$ day.

Based on his initial confusion, Craig now attempts to give a revised version of the paradox showing that an infinite past is impossible:

> But let us turn the story about: suppose Tristram Shandy has been writing from eternity past at the rate of one day per year. Would he now be penning his final page? Here we discern the bankruptcy of the Principle of Correspondence in the world of the real. For according to that principle, Russell's conclusion would be correct: a one-to-one correspondence between days and years could be established so that given an actual infinite number of years, the book will be complete. But such a result is clearly ridiculous. (33)

Craig here takes his misunderstanding of Russell -- his assumption that, on the standard Tristram Shandy paradox, there will be some day on which the autobiography is complete -- and uses it as the basis for a new argument. For if, as he believes, there is a day of completion, then by putting the beginning of the project infinitely far in the past, we can conclude that today is the day of completion. But now absurdity threatens, for why should today be the day of completion, and not yesterday, since even then Shandy had been writing infinitely long, or the day before that, or etc.

A number of the confusions in the 'inverse Tristram Shandy' paradox have been clarified in the literature. Thus the importance of the distinction between the order type ω and the regression type ω^* has been observed, as had the inadequacy of Craig's condition that there be a one-to-one correspondence between days and years, and Craig's assertion that "the Tristram Shandy paradox serves to vindicate G. J. Whitrow's contention that an infinite past entails infinitely distant events" (Craig (1991), 102) has been thoroughly rebutted.[6] Nevertheless, I feel that the central issue here has been missed.

Craig here again appeals to a supposed mismatch between the mathematical and physical worlds. The Principle of Correspondence, supposedly, is shown to fail in "the world of the real". Let us thus consider a mathematical analog to the inverse Tristram Shandy paradox. Let f be a function defined on the non-positive integers, where f(n) gives the day about which Tristram Shandy is writing on day n. Assume that the current day is 0, and the past days correspond with the negative integers in the obvious way. To match the described scenario, two restrictions are necessary. First, if $n-m = 365$, then $f(n) – f(m) = 1$. Second, for all n, $f(n) < n$. However, it is perfectly straightforward to see that no such function is definable. Let $f(0) = k$ for some k. The it follows immediately that $f(365k) = 0$, contradicting the second condition. Craig, then, is quite right to think that the infinitely past Tristram Shandy autobiographical project is impossible – but its impossibility is mathematical, not physical.

The difficulty, of course, is that if the past is infinite Tristram Shandy had no starting point for his autobiography from which to fall behind. Thus if he is, at any given point in time, some finite distance behind, we can project backward and find the point at which he is exactly caught up. This shows not an absurdity with actual infinities, or in particular with an infinite past, but rather with the particular project Tristram Shandy is engaged in. Similar problems could be created for an infinite future. Imagine, for example, that a prophetic Tristram Shandy right now starts to write a prediction of what will happen in the future, starting with the events of some day k days from now. He, as usual, takes one year to write out the events of a single day. Now imagine that this new Tristram continues in his prophetic vein throughout future history. Of course, such a thing cannot happen, since his current time will eventually catch up with his prophetic time. Again, what is shown to be impossible is not an infinite future, but the successful pursuit of certain projects during that future.

§6 Actual Infinities and Cumulative Addition

Craig takes the inverse Tristram Shandy paradox to be one method of drawing out the general impossibility of the creation of actual infinities through successive addition. Craig gives two reasons for thinking that such creation is impossible, First, he notes that "for every element one adds, one can always add one more. Therefore, one can never

[6] See, e.g., Smith (1987), Smith (1993) and Oppy (1995).

arrive at infinity" (31). This argument, however, rests on a confusion. Of course it is true that one can always add one more, but the possibility of adding one more to a given collection does not show that that collection is not an actual infinity. If it did, Cantor's entire project of transfinite mathematics would be immediately defeated. Craig here seems to be relying on his earlier arguments that one cannot add a book to the infinite library -- arguments which we have now seen to be seriously flawed.

Perhaps Craig means only that adding one to any finite collection results again in a finite collection, and thus that successive addition will never convert a finite collection into an infinite collection. This interpretation is bolstered by his second argument against the creation of actual infinities through successive addition. Craig observes "that $\aleph_0$ has no immediate predecessor. Therefore, one can never reach $\aleph_0$ by successive addition or counting, since this would involve passing through an immediate predecessor to $\aleph_0$." (31). As usual, to the extent that Craig here merely endorses the conclusions of mathematics, I have no quarrel with him. It is certainly true that adding one to any finite number leaves one again with a finite number, and it is equally, and consequently, true that adding one physical object to any finite collection of physical objects leaves one again with a finite collection of physical objects.

Nevertheless, mathematics does tell us that in some sense actual infinities can be created through successive addition. The natural numbers form an actually infinite collection, and they are generated through successive application of the successor relation to zero. Denial of the Two-Worlds Principle, then, compels us to accept that actual infinities can be generated in the real world through successive addition in this same sense. Consider the following example: assume that God by fiat creates an actually infinite collection of objects. He does so *en masse* to avoid worries about whether even God can create actual infinities by successive addition; Craig's earlier arguments against the mere existence of an actually infinite collection of physical objects being defeated there is no objection to such *en masse* creation. The objects created are, in particular, a sequence of flags placed in a straight row between two lines (a 'starting line' and a 'finish line') one meter apart. The first flag is halfway between the two lines, the second is halfway between the first flag and the finish line; the third is halfway between the second and the finish line, and so on.[7] Now assume that a runner runs from the starting line to the finish line. For any number n, call the action of the runner passing the $n^{\underline{th}}$ flag A_n.

The runner will thus perform an actually infinite number of actions. Moreover, he will perform them successively, one after the other. Thus his successive performance of

[7] Assume, just to avoid worries about space, that the first flag is one centimeter in width, and that each successive flag is half the width of its predecessor.

individual actions leads eventually to his performance of an infinite number of actions -- an actual infinity has been created by successive addition. Nonetheless, it remains true that no particular action marks the passage from a finite collection of actions performed to an infinite collection of actions performed. We thus distinguish two senses in which an actually infinite collection can be created through successive addition:

(A) An actually infinite collection is created particularly through successive addition if some act of addition creates an infinite collection out of a finite collection.

(B) An actually infinite collection is created procedurally through successive addition if the process of successive addition, carried through to completion, creates an infinite collection.

Particular creation of actual infinities through successive addition is trivially impossible; procedural creation of actual infinities is possible mathematically and consequently also possible physically.[8]

§7 Conclusion

Consideration of the particular difficulties that Craig sees for a physical realization of an actual infinity makes it clear that there is no *distinctive characteristic* of the physical that Craig is appealing to to explain the local violation of the principle (MA) of mathematical application. Something like the Two Worlds Principle is thus required to make sense of his method of argument. But when we consider his arguments with an eye to understanding why there should be this mismatch between the mathematical possibility and the physical impossibility of the actually infinite, we discover at every stage that the bond between math and reality is too tightly woven to allow separation. If Craig wants to reject the potential use of actual infinities in the physical world, he must also reject their use in the mathematical world.

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[8] That there can be a distinction between what can be accomplished by a particular act in a task and what can be accomplished the whole task is a familiar thought from other areas. Thus, in the area of vagueness, the addition of no particular grain of sand transforms a non-heap into a heap, even though the process of adding all the grains does effect that transformation. Or, in the area of emergent properties, the firing of no particular neuron transforms the non-mental into the mental, even though the process of all the firings does effect that transformation.

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